An Order Level Inventory Model under L_2- System with Exponentially Increasing Demand

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ABSTRACT

This paper deals with a deterministic inventory model developed for a single item having two levels of storage. Here the demand is assumed to be exponentially increasing demand. The replenishment rate is infinite and the model is constructed with shortages which are fully backlogged. The problem is to determine the optimal value of S which minimizes the sum of holding cost and shortage cost. Finite horizon models are also studied and the solution of the models was illustrated with the help of a numerical example.

Keywords - **Two levels of storage, Own Warehouse (OW), Rented Warehouse(RW),Exponential increasing demand, Optimal cost.**

I. INTRODUCTION

The problems on classical inventory models which are found in the existing literature generally deal with single storage facility. But when optimal lot size becomes more than the total amount that can be stored in the existing storage facility (Warehouse owned by the management OW) the question of acquiring some extra storage facility to store these excess quantity arises. This additional storage facility may be a rented warehouse (RW) with sophisticated preservation facility and abundant space. Usually, the holding cost of RW is higher than the holding cost of OW. So the goods kept in a RW must be depleted at the earliest. Such inventory model with double storage facility OW and RW was first developed by Hartley [1]. This aspect was further studied in detail by Sarma [2] and denoted the models with two levels of storage as "L_2 – models".

The classical inventory models assume that the demand during an inventory cycle occurs at a constant rate so that on-hand inventory at any time can be represented by a linear function of time. This assumption does not necessarily hold good in many practical situations and the on-hand inventory behaves as a non-linear function. Naddor [3] has specified a general class of demand patterns in terms of on-hand inventory I(t) given by $I(t)= S-D(t/T)^{1/n}$ $0< t< T$ Where S is the stock on-hand at the beginning of the period called the order level, T is the length of the period of the cycle, D is the demand size during T and n is a constant called pattern index. The type of demand that leads to the above form of on-hand inventory is known as power pattern demand. Pakkala and Achary [4] developed a model for deteriorating items with two warehouses. Goswami and Chaudhari [5] formulated the models for time-dependent demand. Rao and Sarma [6] dealt L_2-system with power pattern demand.

This paper deals with a single period inventory model having two levels of storage. The problem is considered in the context of two levels of storage when the order level exceeds the storage capacity of the management, a warehouse is hired to accommodate the extra stock. The on-hand inventory is given by $Q(t)= S - ae^{\alpha t}$, $0 < t < t_1$ where S is initial inventory in the beginning of period after clearing shortages, $ae^{\alpha t}$ is exponentially increasing demand with respect to time t. The optimal value of S which minimizes the sum of holding and shortage cost is determined. Here both finite and infinite horizon models along with their sensitivity analysis are developed.

In the following section, the notations used in the development of the model are listed along with the assumptions made.

II. ASSUMPTIONS AND NOTATIONS

The assumptions and notations are given here under.

(a) $$Q(t)=\begin{cases} S-ae^{\alpha t} & 0\le t\le t_1 \text{ is the demand rate at time 't'} \\ 0 & t>t_1 \end{cases}$$

(b) S is initial inventory at the beginning of the period, after clearing shortages (order level).
(c) T is the length of the period, a known constant.
(d) W is the storage capacity of OW.
(e) Z = (S-W) is quantity stored in RW
(f) H and F are Holding costs per unit time in OW and RW respectively such that F>H.
(g) A_1, A_2 are average inventory held in RW and OW respectively.
(h) A is average inventory in both warehouses.
(i) π is Unit cost of each back ordered item per unit time.
(j) t_w is time at which the consumption from OW starts.
(k) t_1 is time at which shortages starts.
(l) B(t) is back order level at t.
(m) C(S) is optimal cost for L_2-system.

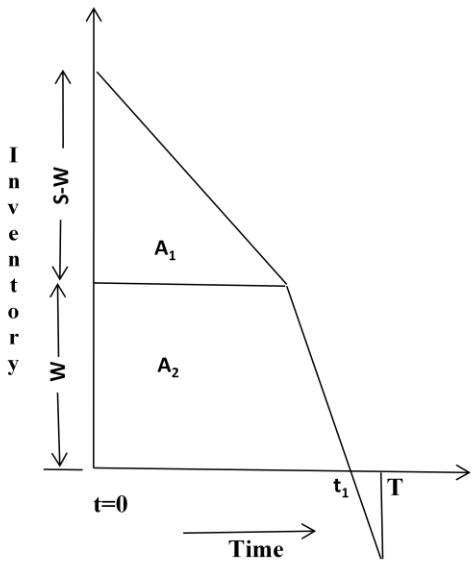
(n) C(W) is optimal cost for S=W .i.e. for L_1-system.
(o) $Q_R(t)$,$Q_0(t)$ are on hand inventory at time t in RW and OW respectively.
(p) L_1, L_2 are systems with one and two levels of storage.
(q) The replenishment rate is infinite and replenishment size is constant.
(r) Lead time is zero and all shortages are backlogged.
(s) The stock kept in OW is used only after exhausting the goods kept in RW.
(t) C_2 is unit shortage cost per unit time.

In the following section the mathematical model to determine the optimal value of S is developed.

III. MODEL DEVELOPMENT AND ANALYSIS

At the beginning of the period, let there be initial stock S. If S< W, it is stored in OW and if S>W, OW is filled up to its capacity and Z= S-W units are stored in RW. Consumption of the stock kept in OW is started only after the RW is completely depleted since holding cost of RW is greater than holding cost of OW. The present inventory situation is shown in the following Fig.1

Figure 1: Mathematical model



The problem is to determine the optimal value of S which minimizes the sum of holding and shortage costs duly considering the model with exponentially increasing demand.

Since shortages are allowed, the system will have positive inventory during the period (0, t) and on-hand inventory is given by $S - ae^{\alpha t}$.

$$Q(t)=\begin{cases} S - ae^{\alpha t}, 0 \le t \le t_1 \\ 0, \text{otherwise} \end{cases} \quad (1)$$

The rented warehouse will hold inventory during (0, t_w) and the on-hand inventory in RW is given by

$$Q_R(t)= Z-ae^{\alpha t} \quad \text{for } 0 \le t \le t_w \quad (2)$$

The average inventory in RW is given by

$$A_1 = \frac{1}{T}\int_0^{t_w} Q_R(t)dt$$

Substituting (2) in the above equation

$$A_1 = \frac{1}{T}\int_0^{t_w}(Z - ae^{\alpha t})dt$$

$$= \frac{1}{T}\left[Zt - \frac{ae^{\alpha t}}{\alpha}\right]_0^{t_w}$$

$$A_1 = \frac{1}{T}\left[Zt_w - \frac{ae^{\alpha t_w}}{\alpha} + \frac{a}{\alpha}\right] \quad (3)$$

$$\because Q_R(t_w) = 0$$

$$\Rightarrow Z = ae^{\alpha t_w} \quad \text{from equation (2)}$$

$$\Rightarrow t_w = \frac{1}{\alpha}\log\frac{Z}{a} \quad (4)$$

Replacing t_w in equation (3)

$$A_1 = \frac{1}{T}\left[Z.\frac{1}{\alpha}\log\frac{Z}{a} - \frac{\cancel{a}}{\alpha}.\frac{Z}{\cancel{a}} + \frac{a}{\alpha}\right]$$

$$A_1 = \frac{1}{T}\left[\frac{Z}{\alpha}\log\frac{Z}{a} - \frac{Z}{a} + \frac{a}{\alpha}\right] \quad (5)$$

Let A denote the average inventory carried in both warehouses then

$$A = \frac{1}{T}\int_0^{t_1} Q(t)dt \quad (6)$$

$$= \frac{1}{T}\int_0^{t_1}(S - ae^{\alpha t})dt$$

$$A = \frac{1}{T}\left[St_1 - \frac{a}{\alpha}e^{\alpha t_1} + \frac{a}{\alpha}\right] \quad (7)$$

$$Q_0(t_1) = 0$$

$$\Rightarrow S = ae^{\alpha t_1} \quad \text{from (1)}$$

$$\Rightarrow t_1 = \frac{1}{\alpha}\log\frac{S}{a} \quad (8)$$

$$A = \frac{1}{T}\left[\frac{S}{\alpha}\log\frac{S}{a} - \frac{S}{\alpha} + \frac{a}{\alpha}\right] \quad (9)$$

Average inventory time units in OW are given by

$$A_2 = A - A_1$$

From equations (5) and (9)

$$A_2 = \frac{1}{T}\left[\frac{S}{\alpha}\log\frac{S}{a} - \frac{S}{\alpha} + \frac{a}{\alpha}\right] - \frac{1}{T}\left[\frac{Z}{\alpha}\log\frac{Z}{a} - \frac{Z}{\alpha} + \frac{a}{\alpha}\right] \quad (10)$$

Back order $\bar{B} = \frac{1}{T}\int_{t_1}^{T}(ae^{\alpha t} - ae^{\alpha t_1})dt$

$$\Rightarrow \overline{B} = \frac{1}{T}\left[\frac{a}{\alpha}e^{\alpha T} - ST - \frac{S}{\alpha} + \frac{S}{\alpha}\log\frac{S}{a}\right] \qquad (11)$$

The sum of inventory holding cost and shortage costs gives the following cost function C(S)

$$C(S) = FA_1 + HA_2 + \pi\overline{B}$$

$$= \frac{F}{\alpha T}\left[(S-W)\log\frac{S-W}{a} - (S-W) + a\right] + \frac{H}{\alpha T}\left[S\log\frac{S}{a} - S -\right.$$
S−WlogS−Wa+S−Wα+πTaeαTα−ST+Sα logSa− Sα
(12)

The optimal value of S which minimize equation (12) is obtained by

$$\frac{\partial C(S)}{\partial S} = 0 \qquad (13)$$

$$\Rightarrow \frac{F}{\alpha T}\left[(s-w).\frac{1}{\frac{S-W}{a}}.\frac{1}{a} + \log\frac{S-W}{a} - \frac{1}{a}\right] + \frac{F}{\alpha T}\left[S.\frac{1}{\frac{S}{a}}.\frac{1}{a} +\right.$$
logSa−1S−W1S−Wa.1a−logS−Wa+1+πT
$$\left[0 - T - \frac{1}{\alpha} + \frac{s}{\alpha}.\frac{1}{s/a}.\frac{1}{a} + \frac{1}{\alpha}\log\frac{s}{a}\right] = 0$$

$$\Rightarrow \frac{F}{\alpha T}\log\frac{S-W}{a} + \frac{H}{\alpha T}\log\frac{S}{a} - \frac{H}{\alpha T}\log\frac{S-W}{a} - \pi - \frac{\pi}{\alpha T} + \frac{\pi}{\alpha T} + \frac{\pi}{\alpha T}\log\frac{S}{a} = 0$$

$$\Rightarrow (F-H)\log\frac{S-W}{a} + (\pi + H)\log\frac{S}{a} = \pi\alpha T \qquad (14)$$

S is the solution of the above transcendental equation which can be solved by numerical methods.

The sufficient condition for S to be minimum is

$$\frac{d^2c}{ds^2} > 0$$

$$\Rightarrow \frac{(F-H)}{S-W} + \frac{H+\pi}{S} > 0$$

The solution of equation (14) gives value of S says S^0. Once S^0 is obtained, its associated cost C(S) is compared with 'threshold cost' i.e. C(W), cost at maximum capacity of OW. If C(S) < C(W) then S^0 is the optimal value of S with the case of L_2-system. Otherwise, the optimal value would be restricted to OW capacity i.e., W which requires OW only.

In the following section the working of above model is illustrated and sensitivity is examined with respect to 'a' and 'α'.

IV. NUMERICAL ILLUSTRATION

Numerical study is carried out on the previous model to find the optimum cost along the following steps.

Step 1. The model is evaluated with illustrative data

F=2, H=1, S=100, a=150, T=1, W=50, α=0.5,π=0.25

Step 2. Substituting the above values in equations (12) and (14), using Newton-Raphson method the value of S^0 is obtained.

Step 3: The value of $S^{0\ is}$ substituted in equation (12) and optimal cost C(S) is obtained.

Step 4. Considering S=W ,substituting the optimal value S^0 in C(W), C(S) is compared with 'threshold cost' i.e. C(W), cost at maximum capacity of OW.

The value of S^0 obtained by above method is S^0= **194.5324 units.**

The value of C(S)= **Rs.238.9356**

The value of C(W)= **Rs.243.59871**

It can be seen that C(S)< C(W) which implies S^0 is the optimal value of S which shows the usage of L_2 system. Otherwise, the optimal value would be restricted to OW capacity i.e. W which requires OW only.

V. SENSITIVITY ANALYSIS

For the inventory system mentioned above , sensitivity analysis is performed to study the changes in the values of the optimal average cost, and demand parameters for different values of W keeping the other system parameters same. The results are tabulated in the Table 1and Table 2. They are also displayed graphically in the Fig.2 and Fig.3 respectively.

Table 1

Effect of change of 'a' on optimal cost C(S) keeping 'W' fixed

	W=50		W=75		W=100		W=125		W=150		W=175	
a	S	C(S)	S	C(S)	S	C(S)	S	C(S)	S	C(S)	S	C(S)
90	157.12	198.24	166.36	154.11	176.02	111.25	185.81	70.30	194.95	33.47	200.95	10.30
120	178.19	213.03	185.36	171.72	192.55	132.49	199.32	96.85	204.68	68.82	205.83	62.52
150	194.53	238.93	200.10	201.37	205.37	167.05	209.80	138.09	212.23	120.18	209.61	131.26
180	207.88	275.01	212.15	241.69	215.85	212.58	218.36	190.72	218.40	183.18	212.70	211.20
210	219.17	319.83	222.33	290.79	224.71	267.05	225.60	252.34	223.61	255.63	215.32	299.26
240	228.94	372.09	231.15	347.30	232.38	328.90	231.87	321.23	228.13	333.85	217.58	393.52

Table 2

Relation between Shape Parameter (α) & Optimal Cost (C) keeping 'W' fixed

	W=50		W=75		W=100		W=125		W=150		W=175	
α	S	C(S)	S	C(S)	S	C(S)	S	C(S)	S	C(S)	S	C(S)
0.1	187.03	12258.36	193.34	10577.09	199.49	8895.86	204.99	7185.74	208.77	5384.27	207.88	3337.33
0.2	188.91	2715.825	195.03	2339.45	200.96	1966.91	206.21	1595.81	209.63	1222.282	208.31	843.32
0.3	190.78	1038.002	196.72	890.98	202.43	747.687	207.40	609.341	210.50	479.73	208.74	373.96
0.4	192.65	481.6946	198.41	410.82	203.90	343.353	208.60	281.362	211.37	230.24	209.18	208.09
0.5	194.53	238.9356	200.10	201.43	205.37	167.06	209.80	138.092	212.23	120.18	209.61	131.26
0.6	196.40	115.5035	201.79	95.12	206.84	77.65	211.00	65.38	213.10	63.843	210.05	90.06
0.7	198.27	46.557853	203.48	35.90	208.31	27.99	212.20	25.058	213.96	32.387	210.48	65.93
0.8	200.15	5.723046	205.17	0.9961	209.78	-1.11	213.40	1.536	214.83	13.989	210.91	51.25

Figure 2

Graphical display of 'a' and C(S)

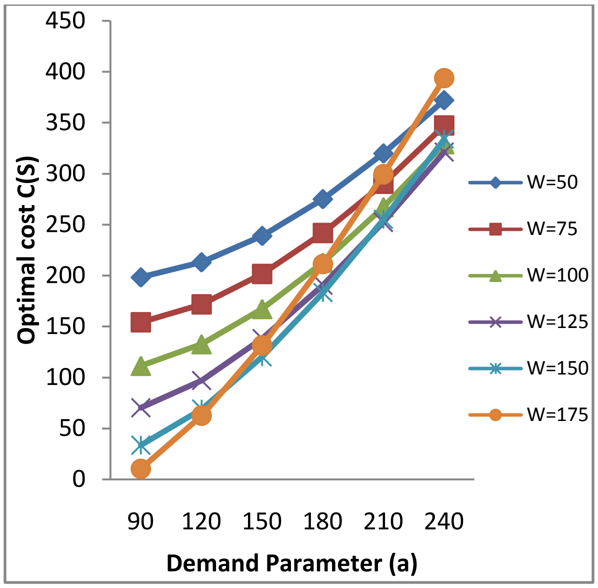
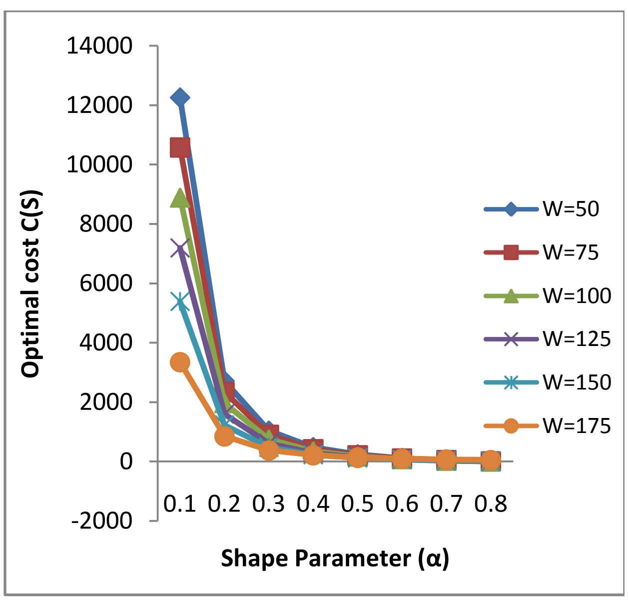


Figure 3

Graphical display of 'α' and C(S)



OBSERVATIONS

The two level inventory model developed above leads to following observations:

- Sensitivity analysis performed on 'a' and corresponding changes in C(S) are presented in Table 1. It is observed that, the cost increases as the value of 'a' is increasing. It is also observed that if value of 'a' is taken less than 80, the L_2 –system is not feasible.
- Corresponding to Table 1, Fig.2 displays the graph of 'a' versus C(S). It is observed that as 'a' increases, the cost is also increasing.
- From Table 2, for different values of 'α', the change in C(S) is presented, keeping 'W' fixed, and this is done for different values of 'W'. It is observed that, there is drastic decrease in value of C(S) as 'α' increases from 0.1 to 0.2 for all values of 'W'. The cost is decreasing as 'α' value is increasing.
- L_2 –system is feasible whenever 'α' values lie between 0.1 and 0.8.
- Corresponding to Table 2, Fig.3 displays the graph of 'α' versus C(S). It is observed that as 'α' is increasing the corresponding cost is decreasing.

DISCUSSION

The discussion made so far is mainly meant for determining order level inventory for an infinite horizon model when the demand increases exponentially. In the subsequent section a finite horizon model is developed when the demand increases exponentially.

VI. FINITE HORIZON MODEL

The length of replenishment period is t_p and is given by

$$t_p = \frac{U - t_1}{m} \qquad (15)$$

Where t_1 is defined as in infinite horizon model equation (8) and 'm' is the number of replenishments. U is the fixed quantity i.e. planning horizon and m be the number of replenishments made during the period ($U-t_1$). Substituting t_1 value in t_p we get

$$t_p = \frac{1}{m}\left[U - \frac{1}{\alpha}\log\frac{S}{a}\right]$$

The total cost of the system during the horizon U is given by

$$(S,m)=mC_2+m(FTA_1+HTA_2+\overline{B}T) \quad (16)$$

Where C_2 is unit shortage cost, F and H are holding cost in OW and RW respectively.

A_1 and A_2 are average inventory held in RW and OW , given by equations (5) and (10) .The number of shortages can be obtained using $\overline{B}$ in the finite horizon model. In the above equations T is used as multiplicand so as to get total number of units carried out in OW and RW respectively.

The total cost of the system during the horizon U is given by

$$K(s,m) = mC_2 + m\{F\left[\frac{z}{a}\log\frac{z}{a} - \frac{z}{\alpha} + \frac{a}{\alpha}\right] + H\left[\left(\frac{s}{\alpha}\log\frac{s}{a} - \frac{s}{\alpha} + \frac{a}{\alpha}\right) - \left(\frac{z}{\alpha}\log\frac{z}{a} - \frac{z}{\alpha} + \frac{a}{\alpha}\right)\right] + \pi\left[\frac{a}{\alpha}e^{\alpha U} - sU - s\alpha + s\alpha\log sa\right\} \quad (17)$$

$$Z = S - W$$

Fixing m to m^* the corresponding optimum values of $S(m^*)$ of 'S' is the solution of

$$\frac{\partial K(S,m^*)}{\partial S} = 0$$

$$\Rightarrow F\log\frac{S-W}{a} + H\log\frac{s}{a} - H\log\frac{S-W}{a} - \pi\alpha U + \pi\log\frac{S}{a} = 0$$

$$\Rightarrow (F-H)\log\frac{S-W}{a} + (H+\pi)\log\frac{s}{a} = \pi\alpha U \quad (18)$$

The above equation doesn't yield to explicit solution. However, an optimal order quantity say S^o of S using Newton-Raphson method is obtained. The optimal number of replenishments 'm_0' is, then the value of m^* that minimizes $K(m^*)$. Since m^* is non-negative integer the necessary condition for $K(m^*)$ to be minimum at $m^*=m_0{}^*$ is

$$\Delta K(m_0^* - 1) \le 0 \le \Delta K(m_0^*) \quad (19)$$

where $\Delta K(m_o^*) = K(m_o^* + 1) - K(m_o^*)$

Using Taylor series expansion form of logarithmic terms and ignoring terms of second and higher order powers of S* the condition for optimality $m^* = m_0^*$ becomes

$$(m_0^* - 1) \le 1 + \frac{F(S^*-W-a)^2}{aC_2\,\alpha} + \frac{H(2S^*W-W^2-2aW)}{aC_2\,\alpha} + \frac{n(S^{*2}-2aS^*+a^2e^{\alpha U}-a\alpha US^*}{aC_2\,\alpha} \le m_0^* \quad (20)$$

VII. NUMERICAL ILLUSTRATION

Numerical study is carried out on the above model to find the optimum cost along the following steps.

Step 1. The model is evaluated with illustrative data

F=2, H=1, S=100, a=150, T=1, W=50, α=0.5,π=0.25, U=2,C_2=8

Step 2.Substituting the above values in equations in (17) and (18), using Newton-Raphson method the optimal value of S i.e. S^0 is obtained.

Step 3. Substituting the value of S^0 in equation (20) the value of m^* is obtained .

Step 4. Substituting the value of S^0 and m^* in equation (17) optimal cost is obtained.

The value of S^0 obtained by above method is S^0= 153.8139.

The value of K**(S, m) =2391.678** and the value of **m^*=20** steps.

VIII. SENSITIVITY ANALYSIS

For the inventory system mentioned above, sensitivity analysis is performed to study the changes in the values of the optimal average cost, and demand parameters for different values of α keeping the other system parameters same. The results are tabulated in the Table 3 are also displayed graphically in Fig.4.

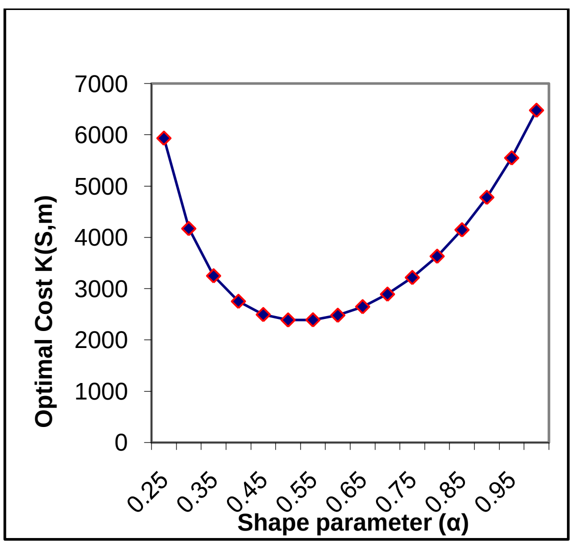
Table 3

Effect of value of 'α' on optimal cost

α	K(S,m)
0.25	5935.954
0.3	4173.773
0.35	3252.524
0.4	2754.954
0.45	2497.301
0.5	2391.611
0.55	2394.081
0.6	2483.343
0.65	2650.549
0.7	2894.562
0.75	3219.576
0.8	3633.946
0.85	4149.695
0.9	4782.417
0.95	5551.451
1.0	6480.247

Figure 4

Graphical display of K(S, m) versus 'α'



OBSERVATIONS

The Fig.4 displays the optimal cost versus shape parameter. The graph between optimal total cost and the shape parameter gives a convex shape which indicates desirable shape parameter to the data under consideration.

IX.CONCLUSION

In this paper, an order level inventory model for exponentially increasing demand under two levels of storage with infinite and finite horizons is presented. This model is applicable in real life situations where there is a limited storage capacity and demand is exponentially increasing. That means where there is a huge demand for the products like electronic goods, essential goods etc, the prudent stockiest would like to opt the Rented Warehouse.

For both finite and infinite horizon models of deterministic demand, cost function was obtained. Though the closed form solution could not be obtained, Newton-Raphson method was used to get the solution. From sensitivity analysis performed it is noted that, when there is huge demand the usage of Rented Warehouse is inevitable.

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